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Abstract
We revisit two classic Thouless–Anderson–Palmer (TAP) studies of the Sherrington–Kirkpatrick model (Bray A J and Moore M A 1980 *J. Phys. C: Solid State Phys.* **13** L469; De Dominicis C and Young A P 1983 *J. Phys. A: Math. Gen.* **16** 2063). By using the Becchi–Rouet–Stora–Tyutin (BRST) supersymmetry, we prove the general equivalence of TAP and replica partition functions, and show that the annealed calculation of the TAP complexity is formally identical to the quenched thermodynamic calculation of the free energy at one step level of replica symmetry breaking. The complexity we obtain by means of the BRST symmetry turns out to be considerably smaller than the previous non-symmetric value.

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1. Introduction

The static properties of mean-field spin glasses have been investigated in the past mainly by means of two different approaches: standard thermodynamics, which through the replica method [1–5] aims to compute the partition function and the equilibrium free energy density of the system, and the Thouless–Anderson–Palmer (TAP) method [6–13], which introduces a mean-field free energy F_{TAP}, function of the set m of local magnetizations, $m \equiv \{m_i\}_{i=1}^{N}$. The local minima of $F_{TAP}(m)$ are identified with the metastable states of the system, which at the mean-field level are well defined. At low temperatures the total number $\mathcal{N}$ of minima of the TAP free energy becomes exponentially large [7, 8, 10], and the density of states is given by

$$\rho(f) \sim e^{N\Sigma(f)}. \tag{1}$$

In this expression $\Sigma(f)$ is the *complexity* of the TAP states with free energy density between f and $f + df$.

The existence of two apparently different methods for the study of the static properties of mean-field spin glasses clearly poses a problem of consistency. Both these methods are in principle correct, and the equivalence between the two approaches should hold in general. A direct argument in its favour is the cavity approach [14, 15]: on the one hand the cavity approach is known to be mathematically equivalent to the replica approach; on the other hand, the first step of the cavity approach consists in clustering the configurations into states which are nothing but the TAP states. From the cavity analysis, one can expect *a priori* the existence of a formal relationship between the replica results computed at a given order k of replica symmetry breaking, and the results of TAP solutions, correctly weighted and with an average computed at the order $k - 1$. Unfortunately, when considering the detailed results published so far, there seemed to exist a discrepancy between the two approaches. For instance, the number of TAP solutions computed directly at the order $k = 0$ (annealed average) in [7], disagrees with the one obtained indirectly through the replica method with one step of replica symmetry breaking (i.e. a $k = 1$ computation) [15]. This apparent discrepancy has led us to give a careful look at this consistency problem. We show here that the TAP approach must be reconsidered: using a supersymmetry, one finds that the two approaches are in fact fully consistent.

From a general point of view, the problem of consistency between the TAP approach and standard thermodynamics can be posed in a twofold way. First, the two methods must agree in the calculation of the thermodynamic quantities that can be computed in both frameworks. This requirement is a strong form of consistency. At a weaker level, regarding objects which are inherently defined only within the TAP method, we may still expect to find some consistency, and even some formal connections, with the standard thermodynamic approach.

The strong consistency basically asks that the equilibrium free energy computed within the two approaches must be the same. Given that F_{TAP} is a function of the local magnetization, it is not obvious how to compute the partition function in this framework. More specifically, due to the large degeneracy of TAP minima, there is the problem of how to weight correctly these states. This problem was solved by De Dominicis and Young [12] (DDY), who proposed to weight each TAP state α with

$$w_\alpha = \exp[-\beta F_{TAP}(m^\alpha)]/Z_{TAP}. \tag{2}$$

The strong equivalence of the TAP approach to the standard thermodynamic approach is therefore encoded in the equality (which should hold in the thermodynamic limit) $\frac{1}{N} \log Z = \frac{1}{N} \log Z_{TAP}$, that is,

$$\frac{1}{N} \log \left(\sum_\sigma \exp[-\beta H(\sigma)] \right) = \frac{1}{N} \log \left(\sum_{\alpha=1}^{\mathcal{N}} \exp[-\beta F_{TAP}(m^\alpha)] \right). \tag{3}$$

Proving relation (3) in general is highly nontrivial, especially because in spin glasses we can only compare averages over the disorder, and thus, from a practical point of view, we need to compare the average of the replicated partition functions, which will typically require some tricky integration over complicated order parameters. This programme has been carried out by DDY for the Sherrington–Kirkpatrick (SK) model [2]. Their conclusion is that relation (3) is indeed verified, provided that the complexity of the *dominant* TAP states at any temperature is zero. This restrictive hypothesis is satisfied in the SK model, but not in other models where the TAP approach is nevertheless used. Thus, it may seem that either condition (3) is not broadly valid, or that a general proof of it must give up the hypothesis used by DDY.

In the present paper, we show that the proof of (3) given by DDY for the SK model is valid with no extra hypothesis. The key point is that the TAP partition function can be written

in an integral form, with an action which is invariant under the Becchi–Rouet–Stora–Tyutin (BRST) supersymmetry [16, 17]. This symmetry provides the mathematical relations needed to prove equation (3), without the need to invoke any further hypothesis on the complexity of the states.

Concerning the weak consistency of the two methods, the key object is the complexity $\Sigma(f)$. Even though there is no obvious way to compute Σ in the standard thermodynamic framework, we may still impose an important consistency condition: at low temperatures (below the static transition) we expect the equilibrium thermodynamic states to have the same free energy density as the lowest TAP states. In other words, we require that at low enough temperatures there must be no static contribution of the metastable states. This condition is encoded in the equation

$$f_{eq} = f_0 \tag{4}$$

where f_{eq} is the equilibrium free energy density computed in the standard thermodynamic framework, and f_0 is defined by the TAP relation $\Sigma(f_0) = 0$. Apart from this minimal requirement, one may ask whether in the TAP calculation of Σ emerge some deeper formal connections with the standard calculation of the free energy. This fact, of course, seems more than likely if the strong consistency condition (3) holds, even though it is in general not obvious. In the p-spin spherical model [18], relation (4) is satisfied [19–22]. Moreover, for this same model it has been shown in [23] that by means of the BRST supersymmetry the saddle-point equations involved in the TAP calculation of Σ at $f = f_0$ become identical to the static equations in the replica approach. In the SK model, however, relation (4) is not easy to check [7–11, 24], since it requires a full replica symmetry breaking (RSB) calculation of the complexity. Moreover, up to now there was no formal connection between the TAP complexity and equilibrium free energy in the SK model.

The fact that the BRST supersymmetry plays a crucial role in the p-spin, suggests that even in the SK model this symmetry may help to prove (4), and discover other formal connections between the two methods. This is what we show in section 4, where we revisit the annealed calculation of the SK complexity of Bray and Moore [7] (BM), and thanks to the BRST supersymmetry show that at $f = f_0$ this calculation becomes equal to the standard static calculation of the free energy, at one step level of replica symmetry breaking. This result comes hardly as a surprise, once the strong consistency (3) of the two methods is proved. However, there is a subtle point: it is not clear *a priori* whether a formal connection between static and TAP approaches is valid only for the correct full RSB solution of the SK model, or if such a connection is preserved at each finite (though approximated) level of replica symmetry breaking. Our result shows that this second scenario is the correct one.

A further, highly nontrivial consistency check of the static and TAP approaches comes from a comparison of the TAP complexity with the complexity computed by means of constrained thermodynamics, which results in the Legendre transform construction of [21]. Within this method the complexity is given by the Legendre transform of the thermodynamic free energy of the system, thus establishing a connection between TAP and static methods at a *generic* value of f. We will prove that, as in the p-spin spherical model, in the SK model the TAP complexity, once the BRST relations are considered, coincides with the complexity of [21].

In section 2 we introduce the BRST supersymmetry in the context of the TAP approach, and derive the BRST relations that will be used in the rest of the paper. In section 2 we show how the BRST relations enforce the DDY proof of (3), without the need of any extra hypothesis. In section 4 we present the BM calculation of the SK complexity, exploiting the BRST supersymmetry, and in section 5 we show the connection between our results and

the Legendre transform method for the computation of the complexity. Finally, we draw our conclusion in section 6.

2. The BRST supersymmetry in the TAP context

In this section we show why the BRST supersymmetry is helpful in the context of the TAP approach. This is just a specific example of the application of the supersymmetric formalism to statistical mechanics, and in particular to the field of disordered systems [25–29]. As we have seen, the TAP free energy F_{TAP} is a function of the local magnetizations m_i, and the minima of this function are identified with the metastable states of the system (TAP states). In the rhs of equation (3) we have a typical example of a sum over different TAP states, as often needed in this approach. In addition to the partition function Z_{TAP}, we may also compute the density of TAP states (1),

$$\rho(f) = \sum_{\alpha=1}^{\mathcal{N}} \delta[F_{\text{TAP}}(m^\alpha) - Nf] \tag{5}$$

which is the quantity one needs for the computation of the complexity $\Sigma(f)$. In (5), as in (3), TAP states are labelled by α, and m^α indicates the corresponding set of local magnetizations. More generally, in the TAP context we always have to deal with expressions of the form

$$R = \sum_{\alpha=1}^{\mathcal{N}} r[F_{\text{TAP}}(m^\alpha)] \tag{6}$$

r being a generic function of F_{TAP}. The quantity R can be written as

$$R = \sum_{\alpha=1}^{\mathcal{N}} \int \prod_i \mathrm{d}m_i \delta(m_i - m_i^\alpha) r[F_{\text{TAP}}(m)]$$
$$= \int \prod_i \mathrm{d}m_i \delta(\partial_i F_{\text{TAP}}(m)) |\det(\partial_i \partial_j F_{\text{TAP}}(m))| r[F_{\text{TAP}}(m)] \tag{7}$$

where the TAP states have been identified with solutions of the TAP equations $\partial_i F_{\text{TAP}}(m) = 0$. In (7) the modulus of the determinant is quite hard to handle, and due to this it is disregarded in most supersymmetric calculations[3]. This approximation is *a priori* unjustified, since without the modulus we are weighting each TAP solution with the sign of the Hessian determinant, with the risk of uncontrolled cancellations. The situation may slightly improve if the function $r(F_{\text{TAP}})$ is peaked only on low values of the free energy F_{TAP}, as it happens in the calculation of Z_{TAP} at low temperatures, and of $\rho(f)$ at low free energies. In these cases, we may hope that at low temperatures (free energies) only minima of F_{TAP} will dominate the integral in (7). Minima have positive-defined Hessian, and thus the modulus becomes redundant. Moreover, the very identification of TAP states with TAP solution is only sensible if these solutions are minima, rather than generic saddles. Yet, the validity of this scenario should be checked carefully, as can be done for the p-spin spherical model, where the effect of disregarding the modulus is completely under control, due to the fact that the free energy distribution of minima and saddles is completely known [30]. In the SK model we do not have such a precise information about the TAP free energy saddle points, and therefore dropping the modulus is quite risky. However, our aim here is to show that all the classic TAP calculations (where

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[3] For an illuminating discussion of the problem of removing the determinant in the supersymmetric formalism see [28]. This same problem was already taken into consideration in [25, 26].

the modulus was disregarded) are formally consistent with the standard static approach, thus justifying *a posteriori* this approximation.

We can use an exponential representation for the δ function and the determinant,

$$\prod_i \delta(\partial_i F_{\text{TAP}}) = \int_{-i\infty}^{+i\infty} \prod_i \frac{dx_i}{2\pi i} \exp\left(\sum_i x_i \partial_i F_{\text{TAP}}(m)\right) \tag{8}$$

$$\det(\partial_i \partial_j F_{\text{TAP}}) = \int_{-\infty}^{+\infty} \prod_i d\bar{\psi}_i \, d\psi_i \exp\left[\sum_{ij} \bar{\psi}_i \psi_j \partial_i \partial_j F_{\text{TAP}}(m)\right] \tag{9}$$

where $\{\bar{\psi}, \psi\}$ are anti-commuting Grassmann variables. In this way we can write [27]

$$R = \int \mathcal{D}m\mathcal{D}x\mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{S(m,x,\bar{\psi},\psi)} \tag{10}$$

where the action S is given by

$$S(m, x, \bar{\psi}, \psi) = \sum_i x_i \partial_i F_{\text{TAP}}(m) + \sum_{ij} \bar{\psi}_i \psi_j \partial_i \partial_j F_{\text{TAP}}(m) + w[F_{\text{TAP}}(m)]. \tag{11}$$

In the case r is an ordinary function we have $w = \log r$, while if $r(F_{\text{TAP}}) = \delta(F_{\text{TAP}})$ then $w = uF_{\text{TAP}}$, where u is an imaginary integration variable to implement the δ function (see the next section). The measures in (10) include the sum over the indices and the constant prefactors. A key property of action (11) is its invariance under a generalization of the Becchi–Rouet–Stora–Tyutin (BRST) supersymmetry [16, 17] (see also [27]): if ϵ is an infinitesimal Grassmann parameter, it is straightforward to verify that (11) is invariant under the following transformation:

$$\delta m_i = \epsilon \psi_i \quad \delta x_i = -\epsilon w' \psi_i \quad \delta \bar{\psi}_i = -\epsilon x_i \quad \delta \psi_i = 0 \quad \Rightarrow \quad \delta S = 0. \tag{12}$$

This generalization of the standard BRST supersymmetry to the case where $w \neq 0$ has been first introduced in [23], in the context of a TAP calculation for the p-spin spherical model. The BRST invariance does not depend on the explicit form of the function $F_{\text{TAP}}(m)$, but simply on the formal structure of action (11). In particular, it is essential that w is a function of the magnetizations m_i only through $F_{\text{TAP}}(m)$. The fact that $\delta S = 0$ under the BRST supersymmetry implies that the average of any observable of the same variables performed with this action must be invariant too. For an observable $\mathcal{O}$,

$$\langle \mathcal{O}(\Lambda) \rangle = \int \mathcal{D}\Lambda \, \mathcal{O}(\Lambda) \, e^{S(\Lambda)} \tag{13}$$

with $\Lambda = \{m, x, \bar{\psi}, \psi\}$, we have

$$\langle \mathcal{O}(\Lambda) \rangle - \langle \mathcal{O}(\Lambda - \delta\Lambda) \rangle = \langle \delta\mathcal{O}(\Lambda) \rangle = 0. \tag{14}$$

This property can be used to generate some useful Ward identities. A fruitful choice is $\mathcal{O} = m_i^k \bar{\psi}_i$ and $\mathcal{O} = x_i^k \bar{\psi}_i$, whose variation gives

$$k\langle m_i^{k-1} \bar{\psi}_i \psi_i \rangle = -\langle m_i^k x_i \rangle \tag{15}$$

$$k\langle x_i^{k-1} w' \bar{\psi}_i \psi_i \rangle = \langle x_i^k x_i \rangle. \tag{16}$$

In particular, the case $k = 1$ gives the BRST equations already used in [23],

$$\langle \bar{\psi}_i \psi_i \rangle = -\langle m_i x_i \rangle \qquad \text{[BRST1]} \tag{17}$$

$$\langle w' \bar{\psi}_i \psi_i \rangle = \langle x_i x_i \rangle \qquad \text{[BRST2]}. \tag{18}$$

The BRST relations are crucial. Thanks to them we will be able to prove in general that any TAP average must give the same result as the equivalent thermodynamic average, and more specifically to reduce the TAP calculation of the complexity to the standard replica calculation of the free energy.

3. The De Dominicis–Young calculation revisited

In 1983, De Dominicis and Young [12] (DDY) introduced the statistical weight (2) for the TAP states and explicitly proved that *under a key hypothesis* the average of the replicated TAP partition function is equal to the average of the replicated standard partition function, i.e.

$$\int \mathcal{D}J \, P(J) Z^n(J) = \int \mathcal{D}J \, P(J) Z^n_{\text{TAP}}(J) \tag{19}$$

which is the practical way to write equation (3). This result is very important, because it proves the identity of the TAP and static approaches *before* any Ansatz for the overlap matrix is done. The hypothesis that DDY invoked to justify their computation is that the complexity of the dominant TAP states at any temperature is zero. More precisely, we can write

$$Z_{\text{TAP}} = \int \mathrm{d}f \, \rho(f) \exp[-\beta N f] = \int \mathrm{d}f \, \exp[-\beta N (f - T\Sigma(f))]$$
$$= \exp[-\beta N (f^\star - T\Sigma(f^\star))] \tag{20}$$

where the dominant free energy $f^\star$ is solution of the equation

$$\beta = \frac{\partial \Sigma(f^\star)}{\partial f}. \tag{21}$$

DDY assumed as a necessary condition that

$$\Sigma(f^\star) = 0. \tag{22}$$

In the present section we argue that although relation (22) is in fact verified in the SK model, it is *not* a necessary condition for the self-consistency of the TAP approach and for its equivalence with the statics.

3.1. The problem and its solution in the DDY formulation

Let us briefly summarize the arguments of DDY leading to condition (22) (in order to keep the notation as clear as possible we treat m as a one-dimensional variable in this section; things do not change with the full N-dimensional representation). First, we introduce an auxiliary magnetic field h both in $F_{\text{TAP}}(m)$ through the term $-hm$, and in the Hamiltonian $H(\sigma)$ with $-h\sigma$. From the very definition of equilibrium magnetization and energy we trivially have

$$\frac{1}{\beta} \frac{\partial Z}{\partial h} = \sum_\sigma \sigma \exp[-\beta H(\sigma)] = \langle \sigma \rangle Z \tag{23}$$

$$-\frac{\partial Z}{\partial \beta} = \sum_\sigma H(\sigma) \exp[-\beta H(\sigma)] = \langle H \rangle Z. \tag{24}$$

On the other hand, from (3) we get

$$\frac{1}{\beta} \frac{\partial Z_{\text{TAP}}}{\partial h} = X - \sum_\alpha^{\mathcal{N}(h,\beta)} \frac{\partial F_{\text{TAP}}(m_\alpha)}{\partial h} \exp[-\beta F_{\text{TAP}}(m_\alpha)] = X + \langle \sigma \rangle Z \tag{25}$$

$$-\frac{\partial Z_{\text{TAP}}}{\partial \beta} = Y + \sum_\alpha^{\mathcal{N}(h,\beta)} \frac{\partial F_{\text{TAP}}(m_\alpha)}{\partial \beta} \exp[-\beta F_{\text{TAP}}(m_\alpha)] = Y + \langle H \rangle Z. \tag{26}$$

To obtain this result we used the following relation,

$$-\frac{\partial F_{\text{TAP}}(m_\alpha)}{\partial h} = \sum_{\sigma \in \alpha} \sigma \exp[-\beta H(\sigma)] \tag{27}$$

where the sum over σ is restricted to those configurations belonging to state α (an analogous relation is valid for $\partial F_{\text{TAP}}/\partial \beta$). On the other hand, in equations (25), (26) X and Y are the contributions coming, respectively, from the dependence on h and β of the total number of TAP states, $\mathcal{N}(h, \beta)$. DDY correctly noted that $X = 0$ and $Y = 0$ are necessary conditions for the consistency of the TAP approach. Moreover, by using (20), they noted that if $\Sigma(f^\star) = 0$, these extra contributions are indeed vanishing in the thermodynamic limit, and thus consistency is recovered. Thus, DDY assumed $\Sigma(f^\star) = 0$ and accordingly imposed $X = Y = 0$ in their computation. More precisely, after averaging over the disorder, the relations $X = Y = 0$ become two equations for the auxiliary fields (equations (32) and (42) of [12]), which DDY used to prove (3). Our point, however, is that the equations $X = Y = 0$ are always satisfied, irrespective of the value of $\Sigma(f^\star)$.

3.2. Role of the supersymmetry

We now show that the mathematical conditions on the auxiliary variables that DDY impose are nothing else than the BRST relations, which are valid in general, and contain no information on $\Sigma(f^\star)$. Let us define

$$G(m, h) = \frac{\partial(\beta F_{\text{TAP}})}{\partial m} \qquad r(m, h) = e^{-\beta F_{\text{TAP}}(m, h)}. \tag{28}$$

We consider relation (23) (a similar reasoning can be easily done also for relation (24)) and use the integral representation of section 2 for Z_{TAP} to obtain

$$\frac{\partial Z_{\text{TAP}}}{\partial h} = -\int \mathrm{d}m \, \mathrm{d}x \, x \, e^{xG(m,h)} \frac{\partial G}{\partial m} r(m, h) + \int \mathrm{d}m \, \mathrm{d}x \, e^{xG(m,h)} \frac{\partial G}{\partial m} \frac{\partial r}{\partial h}. \tag{29}$$

In this expression the first term is X, whereas the second term is just $\langle \sigma \rangle Z$. Integrating by parts we have

$$X = \int \mathrm{d}m \, \delta(G(m, h)) \frac{\partial r}{\partial m}. \tag{30}$$

At this point note that for a generic function $r(m, h)$ there is no need for X to be zero, in much the same way as action (11) is not BRST invariant if r is not a function of F_{TAP}. If, however, $r(h, m) = r(F_{\text{TAP}}(m, h))$, as in the case under examination, we have

$$X = \int \mathrm{d}m \, \delta(G(m, h)) G(m, h) \frac{\partial r}{\partial F_{\text{TAP}}} = 0. \tag{31}$$

Thus, the extra term coming from the differentiation of $\mathcal{N}(h, \beta)$ with respect to h vanishes if the TAP states are weighted with a function of the TAP free energy. A similar conclusion can be easily drawn for differentiation with respect to β, which leads to $Y = 0$. In particular, these relations hold in the case $r(F_{\text{TAP}}) = \exp(-\beta F_{\text{TAP}})$, which is the one analysed by DDY.

The important fact is that from a mathematical point of view, $X = 0, Y = 0$ are consequences of the particular form of action (11), and of its symmetry properties. Indeed, if we express (23) and (24) in the supersymmetric representation (13), we get

$$X = \langle x \rangle$$
$$Y = \left\langle \left[x + \bar{\psi} \psi \frac{\partial}{\partial m} \right] \frac{\partial G}{\partial \beta} \right\rangle = \sum_k c_k \langle m^k x \rangle + k \langle m^{k-1} \bar{\psi} \psi \rangle \tag{32}$$

with

$$c_k = \frac{1}{k!} \frac{\partial^k}{\partial m^k} \frac{\partial G}{\partial \beta}. \tag{33}$$

A comparison between (32) and (15) clearly shows that the relations $X = 0$ and $Y = 0$ are a direct consequence of the BRST symmetry. Thus, the relations imposed by DDY in the calculation of Z^n_{TAP}, and in their demonstration of the equivalence between TAP and static averages, are a consequence of the BRST supersymmetric form of the TAP action (11). These relations are therefore not the expression for the absence of an extensive complexity of the equilibrium states, i.e. they do not imply $\Sigma(f^\star) = 0$. For this reason, the result of DDY is more general than what originally thought. This is consistent with the fact that a similar formal connection between TAP and static approaches is valid in the p-spin spherical model, where, in a certain range of temperature, $\Sigma(f^\star) > 0$.

3.3. Disregarding the modulus

Before closing this section, here is a word of caution on the modulus of the determinant [26, 28, 30]. As already noted above, disregarding the modulus can be very risky, especially if the function $r(F_{\text{TAP}})$ weighting the TAP states in (6) is not peaked on low free energies, and thus minima. As an extreme illustration of this risk we consider the function $r(F_{\text{TAP}}) = 1$. Clearly, from (6) we have

$$R(h) = \mathcal{N}(h) \quad \Rightarrow \quad \frac{\mathrm{d}R}{\mathrm{d}h} = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}h} \neq 0. \tag{34}$$

However, if we use for R the integral representation (7) *and* disregard the modulus, we obtain

$$\frac{\mathrm{d}R}{\mathrm{d}h} = 0 \tag{35}$$

since the BRST supersymmetry is trivially satisfied with $r(F_{\text{TAP}}) = 1$. The problem here is that by disregarding the modulus, and with a flat weight r, we are summing over *all* stationary points of F_{TAP}, each multiplied by the sign of the determinant. The Morse theorem states that this quantity must be a topological constant, only dependent on the manifold over which F_{TAP} is defined and on the boundary conditions on F_{TAP}. Result (35) is therefore correct, but the quantity in this equation is not the same $R(h)$ as in (34). This trivial example shows how important is to weight TAP solutions with a function peaked as much as possible on low free energies when the modulus is disregarded.

4. The Bray–Moore calculation revisited

In the present section we will compute the annealed complexity of the TAP states for the SK model, following closely the classic calculation of Bray and Moore (BM) [7]. Our new contribution will be to exploit the BRST relations (17) and (18) in order to simplify the resulting saddle-point equations. In this way we will prove that the complexity is intimately connected to the 1RSB static free energy.

4.1. General definitions

The TAP free energy for the SK model is given by [6]

$$F_{\text{TAP}}(m) = -\frac{1}{2} \sum_{ij} J_{ij} m_i m_j + \frac{1}{\beta} \sum_i \phi_0(q, m_i) \tag{36}$$

with

$$\phi_0(q, m) = \frac{1}{2}(1 + m) \log\left[\frac{1}{2}(1 + m)\right] + \frac{1}{2}(1 - m) \log\left[\frac{1}{2}(1 - m)\right] - \frac{\beta^2}{4}(1 - q)^2$$

$$= \frac{1}{2} \log(1 - m^2) + m \tanh^{-1}(m) - \log 2 - \frac{\beta^2}{4}(1 - q)^2. \tag{37}$$

The variables m_i are the local magnetizations, and q is the self-overlap of the TAP states,

$$q = \frac{1}{N} \sum_i m_i^2 \tag{38}$$

while the quenched couplings J are random variables with Gaussian distribution,

$$P(J_{ij}) = \sqrt{N/2\pi} \exp\left(-N J_{ij}^2/2\right).$$

The TAP equations and the Hessian of the free energy are respectively,

$$\beta \partial_i F_{\text{TAP}}(m) = -\beta \sum_{j \neq i} J_{ij} m_j + \phi_1(q, m_i) = 0$$

$$\beta \partial_i \partial_j F_{\text{TAP}}(m) = -\beta J_{ij} + \phi_2(q, m_i)\delta_{ij}$$

with

$$\phi_1(q, m) = \beta^2(1 - q)m + \tanh^{-1}(m)$$

$$\phi_2(q, m) = \beta^2(1 - q) + \frac{1}{1 - m^2} + O(1/N). \tag{39}$$

The term of order $1/N$ in $\phi_2(q, m)$ will be dropped in what follows. Following BM we perform an *annealed* calculation of the number of TAP states, i.e. we directly average $\rho(f, \beta|J)$ in (5) over the distribution of the quenched couplings J_{ij},

$$\mathcal{N}(\beta, f) = \int \mathcal{D}J P(J)\rho(f, \beta|J).$$

We use for ρ the integral representation of equations (10) and (11), with

$$\delta(F_{\text{TAP}} - Nf) = \int_{-i\infty}^{+i\infty} \frac{du}{2\pi i} \exp[u(F_{\text{TAP}} - Nf)] \tag{40}$$

and thus $w(F_{\text{TAP}}) = u(F_{\text{TAP}} - Nf)$ and $w' = u$. Thus, the average number of TAP states becomes

$$\mathcal{N}(\beta, f) = \int \mathcal{D}J P(J) \mathcal{D}m \mathcal{D}x \mathcal{D}\bar{\psi} \mathcal{D}\psi \, du \, e^{\beta S(m, x, \bar{\psi}, \psi, u)} \tag{41}$$

where we have multiplied the action by β in order to keep our calculation as close as possible to BM. Before proceeding we mention an important point: in the calculation of BM the J-dependent part of $F_{\text{TAP}}(m)$ in S is eliminated by using the equations $\partial_i F_{\text{TAP}}(m) = 0$, which are enforced by the δ function. More specifically, BM use the equation

$$-\frac{1}{2} \sum_{ij} J_{ij} m_i m_j = -\frac{1}{2\beta} \sum_i m_i \phi_1(q, m_i) \tag{42}$$

which is valid in the TAP states. This substitution simplifies considerably the calculation, but unfortunately the action obtained in this way is no longer BRST invariant. Thus, in the present calculation we must use the full form of $F_{\text{TAP}}(m)$, equation (36), and for this reason some parts of the calculation differ from BM.

4.2. The calculation

The J-dependent part of the action in (41) is given by

$$\beta S_J = -\beta \sum_{ij} J_{ij} \left(x_i m_j + \bar{\psi}_i \psi_j + \frac{1}{2} u\, m_i m_j \right) \tag{43}$$

and after averaging $\mathcal{N}$ over the disorder we obtain the new effective action

$$\beta S = \sum_i x_i \phi_1(q, m_i) + \sum_i \bar{\psi}_i \psi_i \phi_2(q, m_i) + u \sum_i \phi_0(q, m_i) - N\beta u f + \frac{\beta^2 q}{2} \sum_i x_i^2$$
$$+ \frac{\beta^2}{2N} \left(\sum_i m_i x_i \right)^2 - \frac{\beta^2}{2N} \left(\sum_i \bar{\psi}_i \psi_i \right)^2 + N \frac{\beta^2}{4} u^2 q^2 + \beta^2 u q \sum_i m_i x_i. \tag{44}$$

In order to linearize the quadratic terms, we introduce in (41) the following δ functions,

$$\delta \left(qN - \sum_i m_i^2 \right) = \int_{-i\infty}^{+i\infty} \frac{d\lambda}{2\pi i}\, e^{-\lambda q N + \lambda \sum_i m_i^2} \tag{45}$$

$$\delta \left(RN - \sum_i m_i x_i \right) = \int_{-i\infty}^{+i\infty} \frac{dr}{2\pi i}\, e^{-r RN + r \sum_i m_i x_i} \tag{46}$$

$$\delta \left(TN - \sum_i \bar{\psi}_i \psi_i \right) = \int_{-i\infty}^{+i\infty} \frac{dt}{2\pi i}\, e^{-t TN + t \sum_i \bar{\psi}_i \psi_i} \tag{47}$$

and integrate over q, R and T. In this way the integrals in x_i and $(\bar{\psi}_i, \psi_i)$ become Gaussian and can be performed explicitly. The effective action becomes

$$\beta S(\Omega, m) = N\Sigma_0(\Omega) + \sum_i \mathcal{L}(\Omega, m_i) \tag{48}$$

where $\Omega = \{q, \lambda, r, R, t, T, u\}$, and,

$$\Sigma_0(\Omega) = \frac{\beta^2}{2} R^2 - \frac{\beta^2}{2} T^2 - rR - tT - \frac{1}{2} \log(2\pi \beta^2 q) - uf\beta - \lambda q + \frac{\beta^2}{4} u^2 q^2 + \beta^2 u q R$$
$$\mathcal{L}(\Omega, m) = u\phi_0(q, m) - \frac{1}{2\beta^2 q} [\phi_1(q, m) + rm]^2 + \log[\phi_2(q, m) + t] + \lambda m^2. \tag{49}$$

We can now write the average number of TAP states as

$$\overline{\mathcal{N}(\beta, f)} = \int \mathcal{D}\Omega\, e^{N\Sigma_0(\Omega)} \prod_i \int dm_i\, e^{\mathcal{L}(\Omega, m_i)} = \int \mathcal{D}\Omega\, e^{N\Sigma_0(\Omega) + N \log \int dm\, e^{\mathcal{L}(\Omega, m)}}. \tag{50}$$

Thanks to the prefactor N in the exponential, the integral in $\mathcal{D}\Omega$ can be performed with the steepest descent method. In this way we can write the complexity $\Sigma(\beta, f)$ as

$$\Sigma(\beta, f) = \frac{1}{N} \log \overline{\mathcal{N}(\beta, f)} = \Sigma_0(\hat{\Omega}) + \log \int dm\, e^{\mathcal{L}(\hat{\Omega}, m)} \tag{51}$$

where $\hat{\Omega}$ is a solution of the saddle-point equations,

$$\frac{\partial \Sigma_0(\Omega)}{\partial \Omega} + \left\langle\!\!\left\langle \frac{\partial \mathcal{L}(\Omega, m)}{\partial \Omega} \right\rangle\!\!\right\rangle = 0 \tag{52}$$

with

$$\langle\!\langle \mathcal{O}(m) \rangle\!\rangle = \frac{1}{\int dm\, e^{\mathcal{L}(\Omega, m)}} \int dm\, \mathcal{O}(m)\, e^{\mathcal{L}(\Omega, m)}. \tag{53}$$

We can reduce the number of variables by directly solving the saddle-point equations for R and T,

$$\frac{\partial \Sigma_0}{\partial R} = 0 \quad \Rightarrow \quad R = r/\beta^2 - qu$$
$$\frac{\partial \Sigma_0}{\partial T} = 0 \quad \Rightarrow \quad T = -t/\beta^2. \tag{54}$$

In order to have expressions as similar as possible to the ones of BM, we define

$$B = \beta^2(1-q) + t \qquad \Delta = -\beta^2(1-q) - s. \tag{55}$$

Using the explicit forms of ϕ_1, ϕ_2 in (39), and relations (54) and (55), we can rewrite (49) as

$$\Sigma_0(\Omega) = -\lambda q - \beta u f - (B + \Delta)(1-q) + \frac{(B^2 - \Delta^2)}{2\beta^2}$$
$$- \frac{1}{2}\log(2\pi\beta^2 q) - \frac{\beta^2}{4}u^2 q^2 - uq\Delta - \beta^2 uq(1-q) \tag{56}$$

$$\mathcal{L}(\Omega, m) = \log\left(\frac{1}{1-m^2} + B\right) - \frac{[\tanh^{-1}(m) - \Delta m]^2}{2\beta^2 q} + \lambda m^2 + u\phi_0(q, m) \tag{57}$$

where ϕ_0 is given in (37). The remaining saddle-point equations (52) for the variable $\{\lambda, u, B, \Delta, q\}$ are

$$\frac{\partial \Sigma}{\partial \lambda} = 0 \quad \Rightarrow \quad q = \langle\langle m^2 \rangle\rangle \tag{58}$$

$$\frac{\partial \Sigma}{\partial u} = 0 \quad \Rightarrow \quad \beta f = \langle\langle\phi_0(q, m)\rangle\rangle - \frac{\beta^2}{2}uq^2 - q\Delta - \beta^2 q(1-q) \tag{59}$$

$$\frac{\partial \Sigma}{\partial B} = 0 \quad \Rightarrow \quad B\left[1 - \beta^2\left\langle\left\langle\frac{(1-m^2)^2}{1 + B(1-m^2)}\right\rangle\right\rangle\right] = 0 \tag{60}$$

$$\frac{\partial \Sigma}{\partial \Delta} = 0 \quad \Rightarrow \quad \Delta = -\frac{\beta^2}{2}(1-q) + \frac{1}{2q}\langle\langle m\tanh^{-1}(m)\rangle\rangle - \frac{\beta^2}{2}uq \tag{61}$$

$$\frac{\partial \Sigma}{\partial q} = 0 \quad \Rightarrow \quad \lambda = B + \Delta - \frac{1}{2q}\left\{1 - \frac{1}{\beta^2 q}\langle\langle[\tanh^{-1}(m) - \Delta m]^2\rangle\rangle\right\}$$
$$+ u\left\langle\left\langle\frac{\partial\phi_0(q, m)}{\partial q}\right\rangle\right\rangle - \left[\frac{\beta^2}{2}u^2 q + u\Delta + \beta^2 u(1-2q)\right]. \tag{62}$$

These equations, together with relations (56) and (57), can be compared with equations (15)–(17) of BM [7]. The differences are due to the different representation of $F_{\text{TAP}}(m)$ we have taken, in order to preserve the BRST supersymmetry. However, the final result, i.e. the values of Σ_0 and $\mathcal{L}$ in the solution of the saddle-point equations, is exactly the same (in the last of equations (17) of BM, however, there is a term $u\partial f/\partial q$ missing). Of course, if we set $u = 0$, i.e. if we do not impose the constraint on the free energy, our expressions become formally identical to BM.

4.3. Role of the supersymmetry and connection with the statics

Solving the saddle-point equations above, even numerically, is not a simple task. However, the problem becomes much easier if we make use of the two BRST relations (17) and (18).

From (47), (54) and (55), we have

$$\langle \bar{\psi}_i \psi_i \rangle = T = -t/\beta^2 = -B/\beta^2 + (1-q)$$
$$\langle x_i m_i \rangle = R = r/\beta^2 - qu = -\Delta/\beta^2 - (1-q) - qu$$

and thus the first BRST equation becomes

$$B + \Delta = -\beta^2 qu \qquad \text{[BRST1]}. \tag{63}$$

In order to write the second BRST equation we need $\langle x_i x_i \rangle$, which can be computed from equation (44),

$$\langle x_i x_i \rangle = -\frac{1}{\beta^2 q} \left\{ 1 - \frac{1}{\beta^2 q} \langle \langle [\tanh^{-1}(m) - \Delta m]^2 \rangle \rangle \right\}.$$

In this way the second BRST relation becomes (we recall that $w' = u$)

$$\frac{1}{q} \left\{ 1 - \frac{1}{\beta^2 q} \langle \langle [\tanh^{-1}(m) - \Delta m]^2 \rangle \rangle \right\} = u[B - \beta^2(1-q)] \qquad \text{[BRST2]}. \tag{64}$$

Equation (60) admits the solution $B = 0$, and following BM this is the solution we adopt. Setting $B = 0$ into the first BRST relation we obtain

$$\Delta = -\beta^2 qu \tag{65}$$

while substituting equation (37) and the second BRST relation into (62), we find

$$\lambda = \tfrac{1}{2}\beta^2 u^2 q. \tag{66}$$

We now use these two results (65) and (66) to rewrite the complexity $\Sigma(\beta, f)$ in (51). By making the change of variable $m \to h = \tanh^{-1}(m)$ in the integral in (51), and using (56) and (57), we obtain

$$\Sigma(\beta, f) = u \left\{ -\log 2 - \frac{1}{2u} \log(2\pi\beta^2 q) + \frac{\beta^2}{4}[(-u-1)q^2 + 2q - 1] \right.$$
$$\left. + \frac{1}{u} \log \int \mathrm{d}h \, \mathrm{e}^{\mathcal{F}(h;q,u)} \right\} - \beta u f \tag{67}$$

with

$$\mathcal{F}(h; q, u) = -\frac{h^2}{2\beta^2 q} - u \log \cosh h. \tag{68}$$

This form of the complexity can be fully appreciated by recalling the expression for the free energy of the SK model at the one-step level of replica symmetry breaking (1RSB). The 1RSB free energy density is given by [4]

$$\beta F_{\mathrm{1RSB}}(\beta; q_1, x) = -\log 2 + \frac{1}{2x} \log(2\pi\beta^2 q_1) + \frac{\beta^2}{4}[(x-1)q_1^2 + 2q_1 - 1]$$
$$- \frac{1}{x} \log \int \mathrm{d}h \, \mathrm{e}^{\mathcal{F}(h;q_1,-x)} \tag{69}$$

where the self-overlap q_1 and the breaking point x satisfy the saddle-point equations $\partial F_{\mathrm{1RSB}}/\partial q_1 = 0$, $\partial F_{\mathrm{1RSB}}/\partial x = 0$, and where we have set to zero the mutual overlap q_0. Clearly, we see that there is a striking formal correspondence between Σ and F_{1RSB}. Indeed we have

$$\Sigma(\beta, f; q, u) = \beta u[F_{\mathrm{1RSB}}(\beta; q, -u) - f] \tag{70}$$

and,

$$0 = \frac{\partial \Sigma}{\partial q} = \beta u \frac{\partial F_{\mathrm{1RSB}}(\beta; q, -u)}{\partial q} \tag{71}$$

$$0 = \frac{\partial \Sigma}{\partial u} = \beta u \frac{\partial F_{1RSB}(\beta; q, -u)}{\partial u} + \beta[F_{1RSB}(\beta; q, -u) - f]. \tag{72}$$

Therefore the saddle-point equations for Σ and F_{1RSB} coincide, provided that $f = F_{1RSB}(\beta; q_1, x)$. Moreover, for $f = F_{1RSB}$ we trivially have

$$\Sigma[\beta, F_{1RSB}(\beta; q_1, x)] = 0 \tag{73}$$

and therefore $f_0 = F_{1RSB}$: as expected, and anticipated in the introduction, the lowest TAP states have the static free energy density.

The fact that $q_1 = q$ and $x = -u$, at f_0 can also be obtained by directly setting $f = f_0 = F_{1RSB}$ in the saddle-point equations (58)–(62). From (59) and (61), after some algebra, we obtain

$$-\frac{1}{4}\beta^2 q^2 u + \frac{1}{2u}\log(2\pi\beta^2 q) - \frac{1}{u}\log \int dh\, e^{\mathcal{F}(h;q,u)} - \langle\langle \log \cosh h \rangle\rangle = 0 \tag{74}$$

while equation (58) becomes

$$1 + \beta^2 qu[(u+1)q - 1] - \frac{1}{\beta^2 q}\langle\langle h^2 \rangle\rangle = 0 \tag{75}$$

where the averages $\langle\langle \cdot \rangle\rangle$ are now performed with the distribution $\exp[\mathcal{F}(h; q, u)]$ of (68). Once we set $u = -x$ and $q = q_1$ equations (74) and (75) coincide with the 1RSB saddle-point equations, i.e. $\partial F_{1RSB}/\partial x = 0$ and $\partial F_{1RSB}/\partial q_1 = 0$ [4]. Summarizing, the complexity Σ vanishes at the 1RSB free energy density, and at this point the two calculations are formally equivalent, since the saddle-point parameters coincide, and Σ and F_{1RSB} are related by equation (70).

Some considerations are in order. First, this identification of the saddle-point parameters in the two calculations has a clear physical meaning, already discussed in [23]. The self-overlap q_1 of the 1RSB states is just the same as the self-overlap of the TAP states at $f = F_{1RSB}$. Less obvious is the relation $u = -x$, since the 1RSB breaking point x does not seem trivially related to the parameter u. However, we must note that

$$\frac{d\Sigma(f)}{df} = -\beta u(f) \tag{76}$$

and thus $-\beta u(F_{1RSB})$ is the slope of the complexity at the lowest free energy, which is indeed equal to the static breaking point βx.

Secondly, we recall that we have set $q_0 = 0$ in the 1RSB calculation. This is due to the fact that we performed an annealed computation of Σ, and thus we only had one value of the overlap. We believe that a quenched, but replica symmetric, calculation of Σ will be equivalent to the 1RSB static calculation with $q_0 \neq 0$. This brings us to the final point: the TAP method has one less step of replica symmetry breaking as compared to the standard static one. This is simply due to the fact that the elementary objects in the TAP approach are states, while in the static one are configurations. For this reason, a model as the p-spin spherical spin glass, which is exactly solved by a 1RSB Ansatz, has a complexity which is exactly replica symmetric. In the SK model, however, which needs a full RSB static solution, also the correct complexity will need to be computed at a full RSB level [23].

4.4. Numerical solution of the saddle-point equations

We now consider the saddle-point equations for Σ at a generic value of $f \geqslant F_{1RSB}$. We are left with two unknown variables, q and u, and with three unused equations, that is (58), (59)
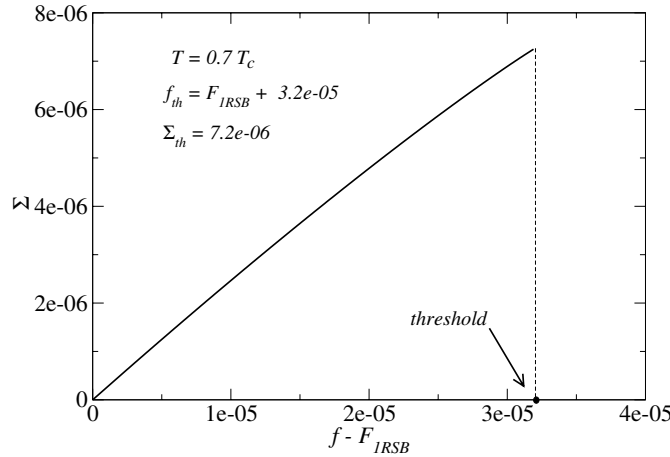
Figure 1. Complexity Σ as a function of the free energy density difference $f - F_{1RSB}(\beta)$, for $T = 0.7T_c$.

and (61). However, by using the BRST relations it is possible to show that actually equations (58) and (61) coincide. To prove this fact, one must use the formula

$$\langle\langle h^2 \rangle\rangle = \beta^2 q - \beta^4 q^2 u + \beta^4 q^2 u(u+1)\langle\langle \tanh^2 h \rangle\rangle \tag{77}$$

in (64), and then substitute this second BRST relation into either (58) or (61). Thus, the two remaining saddle-point equations for q and u, (58) and (59), become

$$q = \langle\langle \tanh^2 h \rangle\rangle$$
$$\beta f + \langle\langle \log\cosh h \rangle\rangle + \frac{\beta^2}{4}[q^2(1+2u) - 2q + 1] + \log 2 = 0 \tag{78}$$

where the formula

$$\tfrac{1}{2}\log(1 - m^2) = -\log\cosh h$$

was used. It is not difficult to solve numerically the two equations above, and this shows how drastic is the simplification of the calculation due to the use of the BRST supersymmetry. In figure 1 we plot Σ as a function of $f - F_{1RSB}(\beta)$, at $T = 0.7T_c$ (we recall that $T_c = 1$). The system of equations (78) have solution only up to a threshold value $f = f_{th}(\beta)$, where the complexity takes its maximum value Σ_{th}. The threshold free energy turns out to be quite close to $F_{1RSB}(\beta)$. An expansion of the equations close to T_c shows that $f_{th} = F_{1RSB} + O(\epsilon^5)$, where $\epsilon = T_c - T$, whereas $q = O(\epsilon)$ and $u = O(\epsilon)$.

An interesting comparison with the result of BM can be done by computing the *total* complexity of TAP solutions, Σ_{tot}. BM obtain this quantity by removing the constraint given by the δ function on the free energy, that is by setting $u = 0$ in the saddle-point equations and in the complexity. The complexity Σ_{tot} that BM find in this way is nontrivial (figure 1 of [7]). If we set $u = 0$ in our equation (67), however, we obtain a trivial result, that is $\Sigma = 0$. This is consistent with our initial remarks on the modulus of the determinant: without the constraint on the free energy we are counting *all* the stationary points of the TAP free energy, each one weighted with the sign of the determinant. This quantity must be a topological invariant due to the Morse theorem, and thus the complexity must vanish. The fact that this happens in our calculation and not in the calculation of BM shows that the BRST symmetry selects the saddle-point solution which preserves Morse's invariance.
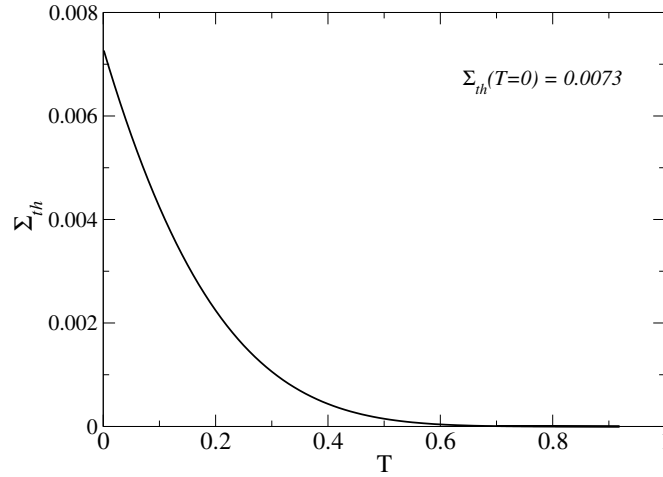
Figure 2. The threshold complexity Σ_{th} as a function of the temperature. Due to the exponential form of the number of states, Σ_{th} is also the total TAP complexity of the system.

In order to obtain Σ_{tot} in our case we simply note that at each temperature the dominant value of the complexity is given by its maximal value, and therefore $\Sigma_{\text{tot}}(T) = \Sigma_{\text{th}}(T)$. Thus, the (annealed) average global number of solutions of the TAP equations is given by $\mathcal{N}_{\text{tot}}(T) \sim \exp(N\Sigma_{\text{th}}(T))$. The threshold values f_{th} and Σ_{th} can be obtained by requiring that the system of equations (78) ceases to have solution, i.e. by imposing a marginality condition on their Hessian. The quantity $\Sigma_{\text{th}}(T)$ is plotted in figure 2: it turns out to be considerably smaller than the non-BRST symmetric value of [7]. This might be one of the reasons why it is so difficult to find solutions of the TAP equations numerically. A numerical analysis of $\Sigma_{\text{tot}}(T)$ shows that it goes to zero for $\epsilon = T_c - T \to 0$ as

$$\Sigma_{\text{tot}}(T) = 0.01\epsilon^6 + \mathrm{O}(\epsilon^7). \tag{79}$$

We conclude this section with an interesting remark on the comparison between the SK and the p-spin spherical model. In the p-spin model when we compute the complexity without the constraint on the free energy (i.e. setting $u = 0$) and without the modulus, we find two saddle points: a BRST symmetric one, which gives $\Sigma = 0$, as in the SK case, and a non-BRST symmetric saddle point, which gives a nontrivial complexity, $\Sigma' \neq 0$. In the p-spin spherical model this second value Σ' of the complexity coincides with Σ_{th}, that is the maximal value of $\Sigma(f)$ obtained by keeping $u \neq 0$ and which is BRST symmetric. The fact that our result, figure 2, does not coincide with the result of BM (figure 1 of [7]) shows that this identity does not hold in the SK model. Thus, the identity $\Sigma' = \Sigma_{\text{th}}$ seems to be a peculiarity of the p-spin spherical model.

4.5. The zero-temperature limit of the total complexity

The largest value of the total complexity $\Sigma_{\text{tot}}(T)$ is achieved at $T = 0$. We find

$$\lim_{T \to 0} \Sigma_{\text{tot}}(T) = 0.0073. \tag{80}$$

This value is remarkably smaller than the value $\Sigma_{\text{tot}}^0 = 0.199$ obtained in [7, 10, 31] by directly solving the zero-temperature limit of the TAP equations. In this section we want to discuss this difference and its implications.
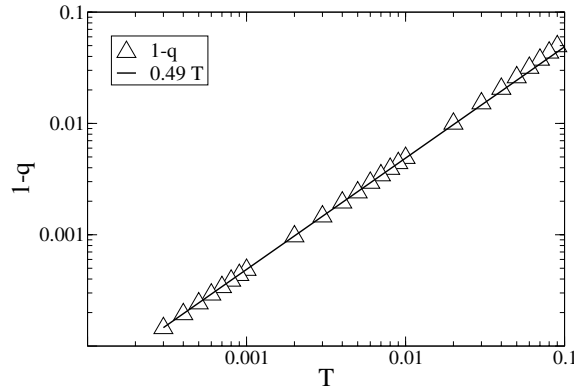
Figure 3. The value of the threshold overlap is used to plot $1 - q$ as a function of T. The full line is the best fit to $1 - q = h_0 T$, with $h_0 = 0.49$.

The TAP equations at finite β are given by

$$m_i = \tanh\{\beta[h_i - \beta(1 - q)m_i]\} \tag{81}$$

where the local field h_i is

$$h_i = \sum_{j \neq i}^{N} J_{ij}m_j. \tag{82}$$

For very large, but finite, β, we can write

$$m_i = \text{sign}[h_i - \beta(1 - q)m_i] \tag{83}$$

where the magnetizations are now spin variable, $m_i = \pm 1$. In the limit $\beta \to \infty$ we have $q \to 1$, and thus we must be careful with the expression in the square bracket. Let us write in general for $\beta \gg 1$,

$$(1 - q) = h_0 \beta^{-\alpha} \qquad \alpha > 0 \tag{84}$$

such that the TAP equations become

$$m_i = \text{sign}\left[h_i - \frac{h_0 m_i}{\beta^{\alpha-1}}\right]. \tag{85}$$

According to the value of α we have different scenarios. For $\alpha < 1$ there is no finite limit of the TAP equations (85). This case is therefore uninteresting. For $\alpha = 1$, on the other hand, we have a well-defined zero-temperature limit of the TAP equations, i.e.

$$m_i = \text{sign}[h_i - h_0 m_i]. \tag{86}$$

This is our case: in figure 3 we show that in the limit $T \to 0$ the overlap q corresponding to the threshold (i.e. total) complexity behaves like

$$q = 1 - h_0 T \qquad h_0 = 0.49 \tag{87}$$

so that $\alpha = 1$. Thus, the zero-temperature limit of the BRST complexity, $\Sigma_{\text{tot}}^0 = 0.0073$, is the complexity of the solutions of equations (86). Note that the T-dependence of the overlap may be different from (87) at values of the free energy different from the threshold.

In the case $\alpha > 1$ the zero-temperature limit of the TAP equations (85) yields

$$m_i = \text{sign}[h_i]. \tag{88}$$

These are the equations considered in [7, 10, 31] in order to compute the zero-temperature complexity (in particular, equations (88) were obtained by assuming that $q = 1 - O(T^2)$,

i.e. $\alpha = 2$). Therefore, the value $\Sigma_{\text{tot}}^0 = 0.0073$ we obtain as zero-T limit of the BRST complexity and the value $\Sigma_{\text{tot}}^0 = 0.199$ obtained in [7, 10, 31], refer to two different sets of equations, that is (86) and (88). The fact that solutions of (86) have a much smaller complexity than those of (88), is due to the fact that the former are actually a subset of the second. Indeed, from equations (86) we have

$$|h_i| > h_0 \tag{89}$$

for all i, which is a very restrictive condition not necessarily satisfied by the solutions of (88).

We note that equations (85) imply

$$m_i h_i > \frac{h_0}{\beta^{\alpha-1}} \tag{90}$$

and thus, given a TAP solution, the change in energy $\Delta E(K)$ when we flip K spins satisfies the relation

$$\Delta E(K) = \sum_i^K m_i h_i - \sum_{ij}^K J_{ij} m_i m_j \sim \sum_i^K m_i h_i - \frac{K}{\sqrt{N}} > K \left(\frac{h_0}{\beta^{\alpha-1}} - \frac{1}{\sqrt{N}} \right). \tag{91}$$

The condition of stability under K spin flips [33] requires $\Delta E(K) > 0$. A sufficient (but not necessary) condition for this to hold is given by

$$\frac{h_0}{\beta^{\alpha-1}} - \frac{1}{\sqrt{N}} > 0. \tag{92}$$

In the case $\alpha = 1$ this condition is trivially satisfied, while for $\alpha > 1$ the order of the two limits $\beta \to \infty$ and $N \to \infty$ becomes important, and the stability under K spin flips may be trickier to prove. Nevertheless, a word of caution on the concept of stability is in order at this point. At finite temperature a stable solution of the TAP equations is a local minimum of the TAP free energy, that is a solution with positive Hessian. Unstable saddle points of the TAP free energy will exist, typically at high free energies. Unfortunately, at $T = 0$ this topological stability is impossible to assess, because of the discrete nature of the model. On the other hand, the K-spin flip stability is easily defined at $T = 0$. However, it is not straightforward to understand what is the relation between these two definitions of stability. We simply note that we have computed the total complexity by finding the *threshold states*: for free energies above the threshold value the saddle-point equations become unstable. Results in continuous mean-field spin glasses (as the p-spin spherical model) suggest that the threshold may indeed be the border between topologically stable solutions, that is TAP minima, and unstable saddle points. It is therefore plausible that our total complexity refers only to TAP minima. The connections we have discovered between complexity and thermodynamics support this hypothesis.

5. Connection with the Legendre transform approach

As discussed in the previous section there is a precise correspondence between the TAP computation at the lower band edge f_0 and the computation of the thermodynamic free energy of the system. However, the relation between the TAP approach and the usual static one (which involves Boltzmann averages) is still deeper: as can be appreciated in (70) there is a formal connection between the TAP complexity at *generic f* and the free energy function F_{1RSB}. These equations also represent an important bridge between the TAP approach and some different methods to compute the complexity which do not rely on TAP equations, but rather on constrained thermodynamics [21, 22]. In [21], it is argued that the complexity $\Sigma(\beta, f)$ of the ergodic components, i.e. states, present at low temperature is given by the Legendre

transform of the free energy $F(\beta, n)$ of n coupled real replicas of the original system. More precisely,

$$\Sigma(\beta, f) = \max_n [\beta n f - \beta F(\beta, n)] \tag{93}$$

where n and f are Legendre conjugated variables, i.e.

$$f = \frac{\partial F(\beta, n)}{\partial n}. \tag{94}$$

If we assume, as is generally accepted, that TAP minima correspond to low-temperature states, then the complexity computed via equation (93) and the complexity obtained from the TAP approach should be equal. In other words, we expect the TAP complexity to satisfy equations (93) and (94).

To establish this point one has first to compute the free energy $F(\beta, n)$ of the coupled system with the replica method and choose an appropriate Ansatz. This problem is discussed for the SK model in [24], where different kinds of Ansatz are analysed: within a generalized two-group Ansatz for $F(\beta, n)$, equation (93) is formally satisfied by the BM solution, as already noted by these authors in [7, 8]. Moreover, a simpler sub-class of two-group saddle-point solutions exists, which corresponds to the following procedure: $F(\beta, n)$ at the k-level of RSB is given by n times the free energy of one system evaluated with $k + 1$ steps of RSB, and breaking parameters $nx_1, nx_2, \ldots, nx_k, n$. Thus, at the lowest possible level of RSB ($k = 0$) one has

$$F(\beta, n) = n F_{1RSB}(\beta; q_1, q_0, n) \tag{95}$$

as also suggested in [21]. With this expression of $F(\beta, n)$, equation (93) has been verified within the TAP approach for models which are exactly solved by 1RSB replica solution [23, 32]. It has also been derived for the Bethe lattice at zero temperature with the cavity method [15], and seems therefore quite robust. Surprisingly, for the SK model this point was still unclear, since the BM solution [7], despite its formal correspondence to a more general two-group static solution, does not appear to satisfy relation (93) with the simpler Ansatz (95). The reason for this can be understood in the light of the previous section: in [7] BM do not use the BRST supersymmetry and consequently consider a larger set of solutions than those physically relevant, much as the generalized two-group Ansatz has a larger set of saddle points than the simpler Ansatz (95). We therefore may expect that once the BRST relations are taken into account, as we have done in the previous section, consistency must be recovered. Indeed, this is precisely what happens: as can be easily verified, once (95) is implemented, equation (70) is equal to (93), with $q_0 = 0$, $q_1 = q$ and $n = -u$. The parameters q_1 and n are variationally fixed: (72) is equal to (94), while variation with respect to q_1 gives back equation (71).

Thus, also for the SK model, we have demonstrated within the TAP approach the validity of relation (93). We note that a crucial element to establish this point is the exploitation of the BRST relations. This is not a surprise since, as discussed in section 3 for equilibrium averages, the BRST supersymmetry is what mathematically guarantees relations of physical significance as (23) and (24).

6. Conclusions

In this paper, we have discussed the equivalence between the TAP approach and the standard thermodynamic method for mean-field spin-glass systems, and have shown that the use of the BRST symmetry of the TAP action is crucial to establish this equivalence.

We demonstrated in section 3 that the BRST relations guarantee the mathematical consistency of the TAP approach and ensure the validity of physically relevant relations. We have shown that the DDY computation, which establishes for the SK model a formal connection between the TAP and the thermodynamic partition functions, is generally valid if the BRST symmetry is taken into consideration, and does not rely on any hypothesis about the nature of the equilibrium states. Furthermore, we have revisited the BM calculation of the complexity for the SK model, with the support of the BRST relations. Thanks to them, we have been able to solve the saddle-point equations at given free energy, we have explicitly shown the equivalence of the lowest TAP solutions with the thermodynamic equilibrium states and we have verified the consistency of the TAP computation of the complexity with the Legendre transform construction.

It should be noted that the BRST relations have a very delicate role due to the presence of quenched disorder. Indeed, once the average over the disorder is taken and the auxiliary variables are introduced, the BRST symmetry of the effective action is much less transparent. In the thermodynamic limit, the auxiliary variables become variational parameters which must satisfy the saddle-point equations. The solutions of these equations correspond to averaged quantities (for example, in the BM calculation $R = \langle x_i m_i \rangle$) and should therefore satisfy the appropriate BRST relations. However, these relations are *not* automatically satisfied by all the saddle-point solutions. In other words, the set of solutions of the saddle-point equations in general also includes solutions which are not BRST invariant. Thus, the BRST relations must be explicitly imposed, providing some additional equations that must be satisfied by the variational parameters. Besides, as we have seen, this procedure provides a great simplification in the computation, since it drastically decreases the number of unknown variables.

Our BRST calculation gives a total complexity at $T = 0$ which is much smaller than the value previously obtained by other classic calculations. We have shown that technically this is due to the fact that the BRST saddle point provides a set of TAP equations at $T = 0$ which are different from the ones previously considered. In particular, the zero-temperature BRST-TAP solutions have local fields larger than a finite value h_0 at all sites, and this condition is responsible for the drastic decrease in the complexity. We have outlined what are the possible physical explanations of this difference, but more study on the $T \to 0$ limit is necessary for a complete clarification of this point.

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